

Unit I Theory of Equation.

Descartes Rule of Sign - Approximate Solutions of Polynomials by
 Horner's method - Newtons Raphson method of solution of a
 Cubic polynomial.

Concept of change of sign in the Coefficients of a Polynomial

Consider the Polynomial,

$$2x^7 - 3x^6 - 4x^5 + 5x^4 + 6x^3 - 7x + 8$$

Let us denote the sign of the coefficients using the symbols '+' and '-'

+, -, -, +, +, -, +

Note: ① There is no symbol (+ or -) corresponding the coefficient x^2 .

② In the above sequence there are four changes in sign

Definition change of sign in the Coefficients

Let $P(x)$ be a polynomial. If the Coefficient of x^{j+1} and the Coefficient of x^j are of different signs, then the change of sign in the Coefficients is occur at the j^{th} power of x

Descartes Rule

If p is the number of positive zeros of a polynomial $P(x)$ with real coefficients and s is the number of sign changes in coefficients of $P(x)$, then $s - p$ is a non negative even integer.

Note: ① The number of positive roots of a polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(x)$.

Note ② The number of negative zeros of the polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(-x)$ and the difference between the number of sign changes in coefficients of $P(-x)$ and the number of negative zeros of the polynomial $P(x)$ is even.

Note ③ As the multiplication of a polynomial $P(x)$ by x^k ($k > 0$), neither changes the number of positive zeros of the polynomial nor the number of sign changes in coefficients.

Attainment of Bounds

① Bounds for the number of real roots

Consider the polynomial $P(x) = (x+i)(x-1)(x-2)(x+i)(x-i)$
has the zeros $-1, 1, 2, -i$ and i .

$P(x)$ can be rewritten as

$$P(x) = x^5 - 2x^4 - x + 2 \quad \rightarrow \textcircled{1}$$

the sign of coefficients as

$$+ \quad - \quad - \quad +$$

The number of sign changes in $P(x)$ is 2

$\therefore P(x)$ has at most 2 positive real ~~roots~~ ^{zeros} (They are 1 and 2) ~~They are~~

Moreover, $P(-x) = (-x)^5 - 2(-x)^4 - (-x) + 2$ $\left\{ \because \text{from } \textcircled{1} \right.$

$$P(-x) = -x^5 - 2x^4 + x + 2$$

The sign of coefficients of $P(-x)$ are

$$- \quad - \quad + \quad +$$

The number of sign changes in $P(-x)$ is 1

The ~~num~~ by descartes rule, the number of negative roots cannot exceed 1.

clearly -1 is the ~~negative~~ negative ~~root~~ zero of $P(x)$ and hence the bounds 2 for positive zeros and the bound 1 for negative zeros are attained.

Note that i and $-i$ are neither positive nor negative zeros, but they are complex zeros.

Example:

W.K.T $(x+2)(x+3)(x+i)(x-i)$ is a polynomial with zeros $-2, -3, -i$ and i respectively.

$$\begin{aligned} \text{let } P(x) &= (x+2)(x+3)(x+i)(x-i) & \int \therefore (a+b)(a-b) &= a^2 - b^2 \\ &= (x^2 + 3x + 2x + 6)(x^2 - i^2) & \therefore i^2 &= -1 \\ &= (x^2 + 5x + 6)(x^2 + 1) & \therefore -i^2 &= 1 \\ &= x^4 + x^2 + 5x^3 + 5x + 6x^2 + 6 \end{aligned}$$

$$P(x) = x^4 + 5x^3 + 7x^2 + 5x + 6 \rightarrow \textcircled{1}$$

The signs of the coefficients of $P(x)$ are,

$+, +, +, +, +$

clearly, the polynomial $P(x)$ has no sign changes $\therefore P(x)$ cannot have more than '0' positive zeros

and $P(-x) = (-x)^4 + 5(-x)^3 + 7(-x)^2 + 5(-x) + 6$ $\int \therefore$ from $\textcircled{1}$

$$P(-x) = x^4 - 5x^3 + 7x^2 - 5x + 6$$

The sign of the coefficient of $P(-x)$ are

$+, -, +, -, +$
 $\underbrace{\hspace{10em}}_{\substack{\textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \textcircled{4}}}$

$P(-x)$ has four sign changes.

By Descartes rule \therefore the number of negative zeros of $P(x)$ cannot have more than 4

Bounds for the number of Imaginary roots (Complex roots)

Let $P(x)$ be a polynomial of degree 'n'.

Let 'm' denote the number of sign changes in coefficients of $P(x)$ of degree n, let k denote the number of sign changes in coefficients of $P(-x)$. Then there exist ~~there~~ at least $n - (m+k)$ imaginary roots for the polynomial $P(x)$.

Example: Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$ has at least six imaginary roots.

Solution

$$\text{Let } P(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$$

Signs of $P(x)$, $+, +, -, -, +$

The number of sign changes in $P(x)$ is 2

\therefore The $P(x)$ cannot have more than 2 positive zeros

$$\begin{aligned}\text{Further } P(-x) &= 9(-x)^9 + 2(-x)^5 - (-x)^4 - 7(-x)^2 + 2 \\ &= -9x^9 - 2x^5 - x^4 - 7x^2 + 2\end{aligned}$$

Signs of $P(-x)$ are;

$-, -, -, -, +$

There is one change in sign for $P(-x)$

and hence the number of negative zeros of $P(x)$ cannot be more than one.

So maximum number of real roots is $2+1=3$

and hence there are at least $9 - (2+1) = 6$ imaginary zeros.
[ie $n - (m+k)$ rule]

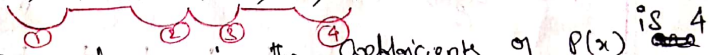
Example (1). Discuss the nature of the zeros of the polynomial

$$3x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 13$$

Solution

Let $P(x) = 3x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 13$

Signs of $P(x)$, +, -, -, +, -, -, +



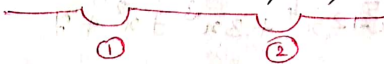
The number of sign changes in the coefficients of $P(x)$ is 4

Hence $P(x)$ cannot have more than 4 positive zeros

and the signs of $P(-x) = 3(-x)^6 - 3(-x)^5 - 5(-x)^4 + 22(-x)^3 - 39(-x)^2 - 39(-x) + 13$

$$P(-x) = 3x^6 + 3x^5 - 5x^4 - 22x^3 - 39x^2 + 39x + 13$$

the signs of $P(-x)$, +, +, -, -, -, +, +



The number of sign changes in the coefficients of $P(-x)$ is 2

Hence $P(x)$ cannot have more than 2 negative zeros.

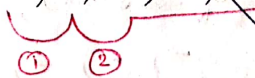
$\therefore P(x)$ can have at most $(4+2) = 6$ real roots.

Example (2). Discuss the nature of the zeros of the polynomial

$$P(x) = x^5 - x^4 + 5x^3 + x^2 + 1$$

Solution

Signs of $P(x)$, +, -, +, +, +



The number of sign changes in $P(x)$ is 2

and $P(-x) = -x^5 - x^4 - 5x^3 + x^2 + 1$

Example (2) Discuss the nature of the zeros of the polynomial,

$$P(x) = x^{18} + 12x^{14} + 2x^{10} + 3x^8 + 5x^2 + 3$$

Solutions

The signs in the coefficients of $P(x)$ are,

+, +, +, +, +, +

The number of sign changes for $P(x)$ is zero

∴ It has no positive roots

And $P(-x) = (-x)^{18} + 12(-x)^{14} + 2(-x)^{10} + 3(-x)^8 + 5(-x)^2 + 3$

$$P(-x) = x^{18} + 12x^{14} + 2x^{10} + 3x^8 + 5x^2 + 3$$

The signs in the coefficients of $P(-x)$ are

+, +, +, +, +, +

The number of sign changes for $P(-x)$ is zero

∴ It has no negative zeros

Thus the polynomial has no real roots and hence all ~~roots~~ ^{zeros} of the polynomial are imaginary zeros.

Example (3) Discuss the nature of the zeros of the polynomial

$$P(x) = x^5 - 12x^4 + 3x^3 + x^2 + 7$$

Solution, The signs of $P(x)$ are

+, -, +, +, +

① ②

The number of sign changes in $P(x)$ is 2

and $P(-x) = (-x)^5 - 12(-x)^4 + 3(-x)^3 + (-x)^2 + 7$

$$P(-x) = -x^5 - 12x^4 - 3x^3 + x^2 + 7$$

The signs of $P(-x)$,

-, -, -, +, +

①

The number of sign changes in $P(-x)$ is 1

⑦ Hence, $P(x)$ has at most 2 positive zeros and at most 1 negative zero.

Since the difference between the number of sign changes in coefficients of $P(-x)$ and the number of negative zeros is even, we cannot have zero negative zeros.

So $P(x)$ can have one negative zero.

Since the difference between number of sign changes in the coefficients of $P(x)$ and the number of positive zeros must be even, we must have ~~zero~~ ~~or~~ either zero or two positive zeros. But as the sum of the coefficients is zero (ie, $1 - 12 + 3 + 1 + 7 = 0$)

$\therefore 1$ is a zero.

Thus we must have ~~two~~ only two positive zeros. Obviously, the other two zeros are imaginary.

Assignment:

① Discuss the maximum possible number of positive and negative zeros of the polynomial

(a) $x^{11} - x^8 + 3x^7 - 2x^6 + x^5 + 4x^3 + 2x^2 + 5x + 1$

(b) $3x^2 - 5x + 8$

(c) $x^2 - 5x + 4$

② Show that the equation $x^9 - 3x^7 + x^4 + 3x^2 + 8 = 0$ has at least six imaginary roots

③ Determine the number of positive roots and negative roots of the equation $x^9 - 3x^6 - 4x^3 = 0$

④ Find the exact number of real zeros and imaginary zeros of the polynomial $3x^9 + x^7 + x^5 + 3x^3 + 7x$