Skill Based Subject

Paper - I

Elementary Mathematics -

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Unit I Theory of Equation.

DesCartes Rule of Sign - Approximate Solutions of Polynomials by Horner's method - Newtons Raphson method of solution of a Cubic polynomial.

Concept of change of sign in the Coefficients of a polynomial

Consider the polynomial,

$$2x^7 - 3x^6 - 4x^5 + 5x^4 + 6x^3 - 7x + 8$$

Let us denote the sign of the coefficients using the Symbols '+' and '-'

+ , - , - , + , + , - , +

Note: ① There is no Symbol (+ or -) corresponding the coefficient x^2.

② In the above sequence there are four changes in sign

Definition change of sign in the Coefficients

Let $P(x)$ be a polynomial. If the coefficient of x^{j+1} and the coefficient of x^j are of different signs, then the change of sign in the coefficients is occur at the j^{th} power of x

Descartes Rule

If p is the number of positive zeros of a polynomial $P(x)$ with real coefficients and s is the number of sign changes in coefficients of $P(x)$, then $s - p$ is a non negative even integer.

Note: ① The number of positive roots of a polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(x)$.

Note ② The number of negative zeros of the polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(-x)$ and the difference between the number of sign changes in coefficients of $P(-x)$ and the number of negative zeros of the polynomial $P(x)$ is even.

Note ③ As the multiplication of a polynomial $P(x)$ by x^k $(k>0)$, neither changes the number of positive zeros of the polynomial nor the number of sign changes in coefficients.

Attainment of Bounds

① Bounds for the number of real roots

Consider the polynomial $P(x) = (x+1)(x-1)(x-2)(x+i)(x-i)$ has the zeros $-1, 1, 2, -i$ and i.

$P(x)$ can be rewritten as

$$P(x) = x^5 - 2x^4 - x + 2 \longrightarrow ①$$

the sign of coefficients as

$$+, -, -, +$$

① ②

The number of sign changes in $P(x)$ is 2

$\therefore$ $P(x)$ has atmost 2 positive real zeros ~~roots~~ (~~They are~~ (They are 1 and 2) ~~Not~~

More over, $P(-x) = (-x)^5 - 2(-x)^4 - (-x) + 2$ {from ①

$$P(-x) = -x^5 - 2x^4 + x + 2$$

The sign of coefficients of $P(-x)$ are

$$-, -, +, +$$

①

The number of sign changes in $P(-x)$ is 1

~~The num~~ By descartes rule, the number of negative roots cannot exceed 1.

Clearly -1 is the ~~negative~~ negative ~~root~~ zero of $P(x)$ and hence the bounds 2 for positive zeros and the bound 1 for negative zeros are attained.

Note that i and $-i$ are neither positive nor negative zeros, but they are complex zeros.

Example:

W.K.T $(x+2)(x+3)(x+i)(x-i)$ is a polynomial with zeros $-2, -3, -i$ and i respectively.

Let $P(x) = (x+2)(x+3)(x+i)(x-i)$ $\{\because (a+b)(a-b) = a^2-b^2$

$= (x^2+3x+2x+6)(x^2-i^2)$ $\quad \because i^2 = -1$

$= (x^2+5x+6)(x^2+1)$ $\quad \because -i^2 = 1$

$= x^4+x^2+5x^3+5x+6x^2+6$

$P(x) = x^4+5x^3+7x^2+5x+6 \longrightarrow (1)$

The signs of the coefficients of $P(x)$ are,

$+, +, +, +, +$

Clearly, the polynomial $P(x)$ has no sign changes
$\therefore$ $P(x)$ cannot have more than '0' positive zeros
and $P(-x) = (-x)^4 + 5(-x)^3 + 7(-x)^2 + 5(-x) + 6$ $\{\because$ from (1)

$P(-x) = x^4 - 5x^3 + 7x^2 - 5x + 6$

The sign of the coefficient of $P(-x)$ are

$$\underbrace{+, -}_{1}\;\underbrace{, +}_{2}\;\underbrace{, -}_{3}\;\underbrace{, +}_{4}$$

$P(-x)$ has four sign changes.
By Descartes rule $\because$ the number of negative zeros of $P(x)$ cannot have more than 4

Bounds for the number of Imaginary roots (Complex roots)

Let $P(x)$ be a Polynomial of degree 'n'.

Let 'm' denote the number of sign changes in Coefficients of $P(x)$ of degree n, let K denote the number of Sign Changes ~~of~~ in coefficients of $P(-x)$. Then ~~there~~ there exist ~~atleast~~ at least $n-(m+K)$ imaginary roots for the Polynomial $P(x)$.

Example: Show that the Polynomial $9x^9+2x^5-x^4-7x^2+2$ has atleast Six imaginary roots.

Solution

Let $P(x) = 9x^9+2x^5-x^4-7x^2+2$

Signs of $P(x)$, $\quad +,\underbrace{+,-}_{1},\underbrace{-,+}_{2}$

The number of sign changes in $P(x)$ ~~of~~ is 2

$\therefore$ The $P(x)$ Cannot have more than 2 Positive zeros

Further $P(-x) = 9(-x)^9+2(-x)^5-(-x)^4-7(-x)^2+2$

$$= -9x^9-2x^5-x^4-7x^2+2$$

Signs of $P(-x)$ are;

$$-,-,-,\underbrace{-,+}_{1}$$

There is one change in sign for $P(-x)$ and hence the number of negative zeros of $P(x)$ Cannot be more than one.

So maximum number of real roots is $2+1=3$ and hence there are atleast $9-(2+1)=6$ imaginary zeros.

[i.e $n-(m+K)$ rule]

Example 1. Discuss the nature of the zeros of the polynomial

$$3x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 13$$

Solution

Let $P(x) = 3x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 13$

Signs of $P(x)$, $+, -, -, +, -, -, +$

The number of sign changes in the coefficients of $P(x)$ is 4

Hence ~~the~~ $P(x)$ cannot have more than 4 positive zeros

and ~~the~~ ~~signs of~~ $P(-x) = 3(-x)^6 - 3(-x)^5 - 5(-x)^4 + 22(-x)^3 - 39(-x)^2 - 39(-x) + 13$

$$P(-x) = 3x^6 + 3x^5 - 5x^4 - 22x^3 - 39x^2 + 39x + 13$$

the signs of $P(-x)$, $+, +, -, -, -, +, +$

The number of sign changes in the coefficients of $P(-x)$ is 2

Hence $P(x)$ cannot have more than 2 negative zeros.

$\therefore$ $P(x)$ can have atmost $(4+2) = 6$ real roots.

Example 2. Discuss the nature of the zeros of the polynomial

$$P(x) = x^5 - x^4 + 5x^3 + x^2 + 1$$

Solution Signs of $P(x)$, $+, -, +, +, +$

The number of sign changes in $P(x)$ is 2

and $P(-x) = -x^5 - x^4 - 5x^3 + x^2 + 1$

Example 2 Discuss the nature of the zeros of the polynomial,

$$P(x) = x^{18} + 12x^{14} + 2x^{10} + 3x^{8} + 5x^{2} + 3$$

Solutions

The signs in the coefficients of $P(x)$ are,

$+, +, +, +, +, +$

The number of sign changes for $P(x)$ is zero

$\therefore$ It has no positive roots

and $P(-x) = (-x)^{18} + 12(-x)^{14} + 2(-x)^{10} + 3(-x)^{8} + 5(-x)^{2} + 3$

$$P(-x) = x^{18} + 12x^{14} + 2x^{10} + 3x^{8} + 5x^{2} + 3$$

The signs in the coefficients of $P(-x)$ are

$+, +, +, +, +, +$

The number of sign changes for $P(-x)$ is zero

$\therefore$ It has no negative zeros.

Thus the polynomial has no real roots and hence all ~~roots~~ zeros of the polynomial are imaginary zeros.

Example 3 Discuss the nature of the zeros of the polynomial

$$P(x) = x^5 - 12x^4 + 3x^3 + x^2 + 7$$

Solution, The signs of $P(x)$ are

$\underbrace{+, -}_{1}\underbrace{\,, +}_{2}, +, +$

The number of sign changes in $P(x)$ is 2

and $P(-x) = (-x)^5 - 12(-x)^4 + 3(-x)^3 + (-x)^2 + 7$

$$P(-x) = -x^5 - 12x^4 - 3x^3 + x^2 + 7$$

The signs of $P(-x)$,

$-, -, \underbrace{-, +}_{1}, +$

The number of sign changes in $P(-x)$ is 1

Hence, $P(x)$ has atmost 2 positive zeros and atmost 1 negative zero.

Since the difference between the number of sign changes in coefficients of $P(-x)$ and the number of negative zeros is even, we cannot have zero negative zeros.

So $P(x)$ can have one negative zero.

Since the difference between number of sign changes in the coefficients of $P(x)$ and the number of positive zeros must be even. we must have either ~~zero~~ zero ~~or~~ or two positive zeros. But as the sum of the coefficients is zero (i.e, $1-12+3+1+7=0$)

$\therefore$ 1 is a zero.

Thus we must have ~~two~~ only two positive zeros.

Obviously, the other two zeros are imaginary.

Assignment:

1. Discuss the maximum possible number of positive and negative zeros of the polynomial (a) $x^{11}-x^8+3x^7-2x^6+x^5+4x^3+2x^2+5x+1$

(b) $3x^2-5x+8$

(c) x^2-5x+4

2. Show that the equation $x^9-3x^7+x^4+3x^2+8=0$ has atleast six imaginary roots

3. Determine the number of positive roots and negative roots of the equation $x^9-3x^6-4x^3=0$

4. Find the exact number of real zeros and imaginary zeros of the polynomial $3x^9+x^7+x^5+3x^3+7x$